

NON-SEPARABLE HILBERT MANIFOLDS OF CONTINUOUS MAPPINGS

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ABSTRACT. Let X, Y be separable metrizable spaces, where X is noncompact and Y is equipped with an admissible complete metric d . We show that the space $C(X, Y)$ of continuous maps from X into Y equipped with the uniform topology is locally homeomorphic to the Hilbert space of weight 2^{\aleph_0} if (1) (Y, d) is an ANRU, a uniform version of ANR and (2) the diameters of components of Y is bounded away from zero. The same conclusion holds for the subspace $C_B(X, Y)$ of bounded maps if Y is a connected complete Riemannian manifold.

1. INTRODUCTION

Most function spaces are infinite-dimensional by nature, and it is natural to ask whether their local structure is similar to that of Hilbert spaces or Banach spaces. From the topological viewpoint, it is known that two Fréchet spaces (i.e., completely metrizable, locally convex real topological vector spaces) are homeomorphic if they have the same weight, due to the characterization of Hilbert spaces by Toruńczyk [16] (cf. [17]). Therefore, any Fréchet space of weight τ is homeomorphic to the *Hilbert space* $\ell^2(\tau)$ of weight τ , in other words, to the Hilbert space with τ orthonormal basis vectors.

Thus we may expect that many function spaces are locally homeomorphic to the Hilbert space $\ell^2(\tau)$ for a suitable τ . Such spaces are called topological *Hilbert manifolds*, or more precisely, $\ell^2(\tau)$ -*manifolds*.

Using the Toruńczyk's characterization mentioned above, many function spaces are known to be Hilbert manifolds. One typical example is the following: the function space $C(X, Y)$ of continuous maps from an infinite compact metric space X into a separable complete metric ANR Y with no isolated points is an ℓ^2 -manifold, with respect to the uniform topology (Sakai [14]).¹

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¹In this case, the uniform topology coincides with many other interesting topologies including the compact-open topology, since X is compact.

In this paper, we prove corresponding results for $C(X, Y)$ with the uniform topology when X is a *noncompact* space, with suitable assumption added to Y . The most significant change from the previous setting is that the function space $C(X, Y)$ becomes non-separable.²

When we apply Toruńczyk’s characterization to prove a space to be a Hilbert manifold, we have first to prove that the space is an ANR, and this is often difficult. To avoid this problem, we assume that the metric space Y is an *ANRU*, which is a uniform version of ANR. The definition of this notion is obtained from that of the usual ANR by replacing all of “continuous maps” and “neighborhoods” by uniform ones, namely, uniformly continuous maps and uniform neighborhoods (see §2.2 for the precise definition). This notion apparently depends on the metric of Y , but in fact depends only on the uniformity it defines (Corollary 2.5). The assumption that Y is an ANRU is not too strong, since all compact polyhedra as well as Euclidean spaces with the usual metric are ANRU’s (Example 2.7).

Now the main theorem of this paper can be stated:

Theorem 1.1. *Let X be a separable noncompact metrizable space and (Y, d) a separable complete ANRU. Assume that the diameters of path-components of Y are bounded away from zero: that is, the set $\{\text{diam } C; C \text{ is a path-component of } Y\}$ has a positive lower bound. Then, the space $C(X, Y)$ of continuous maps from X into Y is an $\ell^2(2^{\aleph_0})$ -manifold with respect to the uniform topology.*

This theorem will be proved in §3.2. The condition on the path-components of Y may seem to be complicated, but this is essential as shown in §4.1 (Theorem 4.1, Corollary 4.2).

Notice that the set $C_B(X, Y)$ of all bounded continuous maps is an open subset of $C(X, Y)$. From examples of ANRU’s mentioned above, we have the following:

Corollary 1.2. *Let X be a separable noncompact metrizable space and let Y be a compact polyhedron without isolated points (equipped with any admissible metric) or the Euclidean space \mathbb{R}^n ($1 \leq n < \infty$). Then, $C(X, Y)$ and $C_B(X, Y)$ are $\ell^2(2^{\aleph_0})$ -manifolds. \square*

This corollary gives many concrete examples of Y for which $C(X, Y)$ is a Hilbert manifold for all noncompact separable metrizable space X . It can also be regarded as a “manifold version” of an elementary fact

² In this case, the compact-open topology is coarser than the uniform topology, and with this topology, $C(X, Y)$ need not have nice local behavior. This problem is discussed in [15].

that the space $C_B(X, \mathbb{R})$ is a Banach space of weight 2^{\aleph_0} , which is homeomorphic to $\ell^2(2^{\aleph_0})$.

The next corollary is concerned with the case where the range space is a Riemannian manifold:

Corollary 1.3. *Let X be a separable noncompact metrizable space, and assume that Y is a connected, complete Riemannian manifold (without boundary) equipped with the geodesic metric. Then $C_B(X, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold.*

The proof of this corollary uses a variation of the main theorem (Corollary 4.3) and will be in §5. This corollary would be false if one replaces $C_B(X, Y)$ by $C(X, Y)$ (Example 5.3). To prove our main Theorem 1.1, according to Toruńczyk's characterization, we have to show that $C(X, Y)$ has a certain general position property. This constitutes the main part of this paper (§3).

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2. PRELIMINARIES

2.1. Basic terminology. In this paper, *spaces* are Hausdorff and *maps* are continuous, while *functions* are not necessarily continuous.

Let X be a space and (Y, d) a metric space. We denote by $C(X, Y)$ the space of all maps from X into Y with the uniform topology.

We follow the standard notation $I = [0, 1]$ and we define $\mathbb{N} = \{1, 2, \dots\}$. Also, by $B(y, r)$ and $\bar{B}(y, r)$ we mean the open ball and the closed ball. For $A \subset Y$ and $r > 0$, we define $N(A, r) = \{x \in X; d(x, A) < r\}$ and $\bar{N}(A, r) = \{x \in X; d(x, A) \leq r\}$.

Let $(X_1, d_1), (X_2, d_2)$ be metric spaces. The *product metric* on $X_1 \times X_2$ is defined by $d((x_1, x_2), (x'_1, x'_2)) = d_1(x_1, x'_1) + d_2(x_2, x'_2)$. Product spaces are assumed to have this metric and the uniformity it induces.

We will frequently use simplicial complexes, which are not assumed to be finite-dimensional or locally finite. The readers are assumed to be familiar with the basic notions concerning simplicial complexes, especially subdivisions. A simplicial complex is denoted by a symbol such as K , which is thought of as a collection of simplexes. The union of

these simplexes, or the *polyhedron* of K , is denoted by $|K|$ and assumed to have the Whitehead weak topology.

We say that a function $\delta: (0, \infty) \rightarrow (0, \infty)$ is a *modulus of continuity*, or a *modulus* for short if δ is monotone and $\lim_{\varepsilon \rightarrow +0} \delta(\varepsilon) = 0$. A typical example of a modulus arises from a uniformly continuous map. Let $f: X \rightarrow Y$ be a uniformly continuous map between metric spaces. Then

$$\delta(\varepsilon) = \sup\{\delta \in (0, 1]; d(f(x), f(x')) < \varepsilon \text{ whenever } d(x, x') < \delta\}$$

defines a modulus δ . We note the following:

Lemma 2.1. *For every modulus δ , there exists a continuous, strictly monotone modulus δ' satisfying $\delta'(\varepsilon) \leq \min\{1, \delta(\varepsilon)\}$ for every $\varepsilon > 0$.*

Proof. We can choose a strictly decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers converging to 0 such that $\delta(\varepsilon_{n+1}) < \delta(\varepsilon_n) < 1$ for each $n \in \mathbb{N}$. Define $\delta': (0, \infty) \rightarrow (0, \infty)$ by

$$\delta'(\varepsilon) = \begin{cases} (1-t)\delta(\varepsilon_{n+2}) + t\delta(\varepsilon_{n+1}) & \text{if } \varepsilon = (1-t)\varepsilon_{n+1} + t\varepsilon_n, t \in I, \\ \varphi(\varepsilon) & \text{if } \varepsilon \geq \varepsilon_1, \end{cases}$$

where φ is an arbitrary strictly monotone continuous function with $\varphi(\varepsilon_1) = \delta(\varepsilon_2)$ and $\varphi(\varepsilon) < 1$ for each $\varepsilon \geq \varepsilon_1$. Then δ' is continuous, strictly monotone and $\delta' \leq \min\{1, \delta\}$. \square

2.2. Preliminaries on ANRU. We introduce a uniform version of the notions of ANE and ANR, namely ANEU and ANRU, but there seems to be several possible definitions for such notions different from each other (see e.g. Isbell [8] and Nguyen To Nhu [12]). We follow [12], which is more convenient in our context of metric spaces. We say that a neighborhood V of a subset A of a metric space is a *uniform neighborhood* if we have $N(A, \varepsilon) \subset V$ for some $\varepsilon > 0$.

Definition 2.2 (ANEU). A metric space X is an *ANEU* if, for every uniformly continuous map $f: A \rightarrow X$ from a closed set A in a metric space Y , there exist a uniform neighborhood V of A and a uniformly continuous extension $\tilde{f}: V \rightarrow X$.

Definition 2.3 (ANRU). A metric space X is an *ANRU* if, for every isometric closed embedding $i: X \rightarrow Y$ into a metric space Y , there exist a uniform neighborhood V of $i(X)$ and a uniformly continuous retraction from V onto $i(X)$.

We see that every ANRU is an ANR by a standard argument as follows: if an ANRU Y is given, we can isometrically embed it as a closed set of a normed space E (see Bessaga-Pełczyński [4, Chapter

II, Corollary 1.1]). Since Y is an ANRU, there is an open (uniform) neighborhood U of Y in E which retracts onto Y . Since U is an ANR as an open set of a normed space, so is Y .

It is easy to see that every ANEU is an ANRU. A less obvious fact is that the converse holds:

Proposition 2.4. *A metric space (X, d) is an ANRU if and only if it is an ANEU.*

In fact, we need only the notion of ANRU and do not need ANEU. However, we include the proof of Proposition 2.4 in Appendix of this paper, since this equivalence implies the uniform invariance of the notion of ANRU, which is sometimes used in this paper:

Corollary 2.5. *If (X, d) and (X', d') are uniformly equivalent metric spaces and (X, d) is an ANRU, then so is (X', d') .*

Indeed, it is clear from the definition that the notion of ANEU is invariant under uniform equivalences (note that it is not so clear for ANRU). Then, this corollary follows from Proposition 2.4.

Now we shall give examples of ANRU's. A *uniform complex* is the polyhedron $|K|$ of a simplicial complex K , endowed with the metric

$$d(x, y) = \sup_{v \in K^{(0)}} |x(v) - y(v)|,$$

where $x(v)$ denotes the barycentric coordinates of x . Notice that the topology induced by this metric is different from the Whitehead weak topology unless K is locally finite. The following fact is known:

Proposition 2.6 (Isbell [7], Theorem 1.9). *Every finite-dimensional uniform complex is an ANRU.* \square

The next examples of ANRU's show that the main Theorem 1.1 implies Corollary 1.2:

Example 2.7. The Euclidean space \mathbb{R}^n , with the usual triangulation (first subdivide the space by unit cubes and triangulate their faces inductively, in order of increasing dimension, without introducing new vertices), is an example of a uniform complex. As a uniform complex, \mathbb{R}^n has a metric uniformly equivalent to the usual one. This means, by Corollary 2.5 and Proposition 2.6, that Euclidean space \mathbb{R}^n with the usual metric (or more generally, a metric induced from a norm) is an ANRU.

Similarly, by Corollary 2.5 and Proposition 2.6, any compact polyhedron with a fixed admissible metric is also an ANRU, since all admissible metrics on a compact space are uniformly equivalent. \square

There is an example of non-ANRU homeomorphic to \mathbb{R} :

Example 2.8. Let Y be the image of the closed embedding $i: \mathbb{R} \rightarrow \mathbb{R}^2$ defined by

$$i(x) = \begin{cases} (|x|, 1/x) & |x| \geq 1, \\ (1, x) & |x| \leq 1. \end{cases}$$

We think of Y as a metric space using the usual Euclidean metric on \mathbb{R}^2 . Then Y is not an ANRU. \square

2.3. Characterization of $\ell^2(\tau)$ -manifolds by Toruńczyk. Let τ be an infinite cardinal and Γ a discrete space of cardinality τ . We denote by $\ell^2(\tau)$ the Hilbert space of weight τ , in other words, the Hilbert space whose orthonormal basis consists of τ vectors. A metrizable space X is an $\ell^2(\tau)$ -manifold if every point of X has an open neighborhood homeomorphic to some open set in $\ell^2(\tau)$.

The characterization of $\ell^2(\tau)$ -manifolds obtained by Toruńczyk is the main ingredient of this paper. To state this theorem, it seems to be convenient to introduce the following notion. A map $f: X \rightarrow Y$ is *approximable* by a map g satisfying a certain property P if for each map $\alpha: Y \rightarrow (0, \infty)$ there exists a map $g: X \rightarrow Y$ satisfying P such that $d(f(x), g(x)) < \alpha(f(x))$ for all $x \in X$.

Theorem 2.9 (Toruńczyk [16], cf. [17]). *Let τ be an infinite cardinal and Γ a discrete space of cardinality τ . The space X is an $\ell^2(\tau)$ -manifold if and only if it satisfies the following conditions:*

- (a) *X is a completely metrizable ANR of weight τ .*
- (b) *For each $n \in \mathbb{N}$, every map from $I^n \times \Gamma$ into X is approximable by a map g for which the family $\{g(I^n \times \{\gamma\})\}_{\gamma \in \Gamma}$ is discrete in X .*
- (c) *For each sequence $\{K_n\}_{n \in \mathbb{N}}$ of finite-dimensional simplicial complexes with the cardinality of vertices $\leq \tau$, every map from the topological sum $\bigoplus_{n \in \mathbb{N}} |K_n|$ into X is approximable by a map g for which the family $\{g(|K_n|)\}_{n \in \mathbb{N}}$ is discrete in X .*

2.4. Basic properties of the function space. Here we prove basic properties of the function space $C(X, Y)$ in our main Theorem 1.1.

Lemma 2.10. *Let X and Y be separable metric spaces.*

- (i) *If X is noncompact and Y contains a homeomorphic copy of I , then the weight of $C(X, Y)$ is 2^{\aleph_0} .*
- (ii) *If Y is an ANRU then $C(X, Y)$ is an ANR.*

Consequently, if X and Y satisfy the assumption in Theorem 1.1, then $C(X, Y)$ is an ANR of weight 2^{\aleph_0} .

Proof. (i) By the assumption, X contains a countable closed discrete set X_0 . We think of $\partial I \subset I \subset Y$, where ∂I is the set of two endpoints of I . Then $C(X_0, \partial I)$ is a discrete space and its weight equals the cardinality, which is 2^{\aleph_0} . Consider the subset $A = \{f \in C(X, Y); f(X_0) \subset \partial I\}$. By the Tietze extension theorem, the restriction $A \rightarrow C(X_0, \partial I)$ is surjective. Therefore, the weight of A , hence of $C(X, Y)$, is at least 2^{\aleph_0} . The reverse inequality easily follows from the fact that both X and Y are separable metrizable.

(ii) By (Bessaga-Pełczyński [4, Chapter II, Corollary 1.1]) we can embed Y isometrically into a normed space E as a closed set. Since Y is an ANRU, it has a uniform neighborhood N in E which has a uniformly continuous retraction $r: N \rightarrow Y$. Then the composition of r defines a (continuous) retraction $C(X, N) \rightarrow C(X, Y)$. Notice that $C(X, N)$ is a neighborhood of $C(X, Y)$ in $C(X, E)$, which is an AR as a normed space. Thus $C(X, Y)$ is embedded into an AR as a retract of some neighborhood. This means that $C(X, Y)$ is an ANR.

Finally, let X and Y be as in Theorem 1.1. The hypothesis in (ii) is satisfied and thus $C(X, Y)$ is an ANR. Since Y is a metric space with a nontrivial path-component, it has a homeomorphic copy of I (see Engelking [6, Exercise 6.3.12]). Therefore, the hypothesis of (i) is also satisfied, and hence $C(X, Y)$ is of weight 2^{\aleph_0} . \square

3. PROOF OF THEOREM 1.1

In this section, we fix spaces X and Y as in the hypothesis of Theorem 1.1: X is a separable metrizable noncompact space and (Y, d) is a separable complete ANRU (§2.2) with the diameters of components away from zero. For brevity, we set

$$\mathbf{F} = C(X, Y).$$

By Lemma 2.10, together with the completeness of Y , the function space \mathbf{F} satisfies the condition (a) of the Characterization Theorem 2.9 for $\tau = 2^{\aleph_0}$. The rest of this section is devoted to the proof that \mathbf{F} also satisfies the conditions (b) and (c).

We may assume that Y is bounded, since if we replace d by $\min\{1, d\}$, the metric space Y still has the same properties stated above and the topology of \mathbf{F} is unchanged. Thus, we can define the sup-metric on \mathbf{F} denoted by d , and the topology of \mathbf{F} is the one induced from d .

3.1. Preliminary lemmas for discrete approximations. Hereafter we fix an isometric closed embedding of (Y, d) into a normed space $(E, \|\cdot\|)$, uniform neighborhood N and a uniformly continuous retraction $r: N \rightarrow Y$ (cf. proof of Lemma 2.10).

Without loss of generality, we can make the following assumptions:

- (♯1) Each component of Y has the diameter at least 1,
- (♯2) The uniform neighborhood N contains the 1-neighborhood $N(Y, 1)$ of Y in E .

Indeed, these conditions are satisfied if we multiply some positive constant to d .

Using the uniform continuity of r , we define $\delta_1: (0, \infty) \rightarrow (0, \infty)$ by

$$\delta_1(\varepsilon) = \sup \left\{ \delta \in (0, \varepsilon/2] ; \begin{array}{l} \|r(x) - r(x')\| < \varepsilon/2 \text{ whenever} \\ x, x' \in N \text{ and } \|x - x'\| < \delta \end{array} \right\},$$

Then δ_0 is a modulus (see §2.1) satisfying $\delta_1(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. By Lemma 2.1, we can take a continuous, strictly monotone modulus δ_0 such that $\delta_0 \leq \min\{1, \delta_1\}$. Consequently, we have the following:

Lemma 3.1. *There exists a continuous, strictly monotone modulus δ_0 with the following properties:*

- (i) $\delta_0(\varepsilon) < \min\{1, \varepsilon\}$,
- (ii) $\|r(x) - r(x')\| \leq \varepsilon/2$ whenever $\|x - x'\| < \delta_0(\varepsilon)$, $x, x' \in N$.

□

Recall that a family $\{f_\lambda\}_{\lambda \in \Lambda}$ of maps from a space Z into a metric space (T, ρ) is *equicontinuous* if for every $\varepsilon > 0$ and $z \in Z$, there is a neighborhood U of z such that $f_\lambda(U) \subset B(f_\lambda(z), \varepsilon)$ for all $\lambda \in \Lambda$.

We introduce the following notation: if $F: Z \rightarrow \mathbf{F}$ is a function, we denote by $\check{F}: X \times Z \rightarrow Y$ the function defined by $\check{F}(x, t) = F(t)(x)$ ($x \in X, t \in Z$). Note that if F is continuous, then so is \check{F} . The converse is not true (consider the case $X = [0, \infty)$, $Y = I$, $Z = I$, $F(t)(x) = 1 - (1 + tx)^{-1}$).

Lemma 3.2. *Let δ_0 be a modulus as in Lemma 3.1. Let $D = \{x_n; n \in \mathbb{N}\}$ be a closed discrete set of X and $\{U_n\}_{n \in \mathbb{N}}$ a discrete open collection with $x_n \in U_n$ ($n \in \mathbb{N}$). Suppose that a map $F: Z \rightarrow \mathbf{F}$ together with equicontinuous family $\{\gamma_n\}_{n \in \mathbb{N}}$ of maps $\gamma_n: Z \rightarrow Y$ satisfy the condition*

$$d(\check{F}(x_n, z), \gamma_n(z)) < \delta_0(\varepsilon(z)) \quad (n \in \mathbb{N}, z \in Z).$$

Then for every map $\varepsilon: Z \rightarrow (0, 1]$ there is a map $F': Z \rightarrow \mathbf{F}$ satisfying

- (i) $\check{F}'(x_n, z) = \gamma_n(z)$ ($n \in \mathbb{N}, z \in Z$),
- (ii) $d(F(z), F'(z)) < \varepsilon(z)$ ($z \in Z$),
- (iii) $\check{F}'|_{(X \setminus U) \times Z} = \check{F}|_{(X \setminus U) \times Z}$,

where $U = \bigcup_{n \in \mathbb{N}} U_n$.

Proof. Fix a continuous function $\beta: X \rightarrow I$ with $\beta|_D = 1$ and $\beta|_{X \setminus U} = 0$. Consider a function $F'': Z \rightarrow C(X, N)$ defined by

$$F''(z)(x) = \begin{cases} \check{F}(x, z) + \beta(x)(\gamma_n(z) - \check{F}(x_n, z)) & \text{if } x \in U_n \subset U, \\ \check{F}(x, z) & \text{if } x \in X \setminus U. \end{cases}$$

It is easy to see that $F''(z): X \rightarrow E$ is certainly continuous. To prove $F''(z) \in C(X, N)$, it is enough to check that $F''(z)(x) \in N$ for $x \in U$. Choosing n such that $x \in U_n$, we have

$$\|\check{F}(x_n, z) - \gamma_n(z)\| < \delta_0(\varepsilon(z)) < \varepsilon(z) \leq 1,$$

and hence $F''(z)(x) = \check{F}(x, z) + \beta(x)(\gamma_n(z) - \check{F}(x_n, z)) \in N(Y, 1) \subset N$. Thus $F'': Z \rightarrow C(X, N)$ is defined as a function.

We claim that F'' is continuous. To see this, $z, z' \in Z$ and $x \in X$. First suppose that $x \in U_n$ for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned} & \|F''(z)(x) - F''(z')(x)\| \\ &= \|(\check{F}(x, z) + \beta(x)(\gamma_n(z) - \check{F}(x_n, z))) \\ &\quad - (\check{F}(x, z') + \beta(x)(\gamma_n(z') - \check{F}(x_n, z')))\| \\ &\leq \|\check{F}(x, z) - \check{F}(x, z')\| + \|(\gamma_n(z) - \check{F}(x_n, z)) - (\gamma_n(z') - \check{F}(x_n, z'))\| \\ &\leq 2d(F(z), F(z')) + \|\gamma_n(z) - \gamma_n(z')\|. \end{aligned}$$

On the other hand, if $x \notin U_n$ for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|F''(z)(x) - F''(z')(x)\| &= \|\check{F}(x, z) - \check{F}(x, z')\| \\ &\leq d(F(z), F(z')). \end{aligned}$$

Combining these inequalities and the equicontinuity of $\{\gamma_n\}_{n \in \mathbb{N}}$, we conclude that F'' is continuous.

Since $r: N \rightarrow Y$ is uniformly continuous, it induces a continuous map $r_*: C(X, N) \rightarrow C(X, Y) = \mathbf{F}$. We define $F': Z \rightarrow \mathbf{F}$ by the composition $F' = r_* \circ F''$. Then, F' is continuous and it is easy to check that F' satisfies the conditions (i) and (iii).

Now it remains only to show that F' satisfies (ii). In view of

$$\begin{aligned} \|\beta(x)(\gamma_n(z) - \check{F}(x_n, z))\| &\leq \|\gamma_n(z) - \check{F}(x_n, z)\| \\ &< \delta_0(\varepsilon(z)), \end{aligned}$$

we have, if $x \in U_n$,

$$\begin{aligned} & \|F'(z)(x) - F(z)(x)\| \\ &= \|r(\check{F}(x, z) + \beta(x)(\gamma_n(z) - \check{F}(x_n, z))) - r(\check{F}(x, z))\| \\ &\leq \varepsilon(z)/2, \end{aligned}$$

due to our choice of δ_0 in Lemma 3.1. Therefore, in view of (iii), we have $d(F'(z), F(z)) \leq \varepsilon(z)/2 < \varepsilon(z)$, and hence, F' satisfies (ii). \square

For $\varepsilon > 0$, we say that a subset $S \subset Y$ is ε -discrete if for every two distinct points $p, q \in S$ we have $d(p, q) \geq \varepsilon$, while S is called ε -dense if for every $y \in Y$ there exists $p \in S$ such that $d(y, p) < \varepsilon$, that is, $N(S, \varepsilon) = Y$.

Let us consider the following conditions $P_1(B, R, \varepsilon)$, $P_2(B, R, \varepsilon)$ for $B, R \subset Y$, $\varepsilon > 0$:

- $P_1(B, R, \varepsilon)$: $B \cap R = \emptyset$ and $B \cup R$ is $\varepsilon/4$ -discrete.
- $P_2(B, R, \varepsilon)$: Both B and R are ε -dense.

We shall prove the following:

Lemma 3.3. *Let $0 < \varepsilon < 1$. Then if $P_1(B, R, \varepsilon)$ holds, there exist $B' \supset B$ and $R' \supset R$ satisfying $P_1(B', R', \varepsilon/2)$ and $P_2(B', R', \varepsilon/2)$.*

Using this lemma inductively, we have the following:

Proposition 3.4. *There exist increasing sequences $B(1) \subset B(2) \subset \dots$ and $R(1) \subset R(2) \subset \dots$ of subsets of Y satisfying $P_1(B(i), R(i), 2^{-i})$ and $P_2(B(i), R(i), 2^{-i})$ for each $i \in \mathbb{N}$.* \square

It may be helpful to think of the elements of $B(i)$ and $R(i)$ as “blue points” and “red points”, respectively.

Proof of Lemma 3.3. Let $B' \subset Y$ be a maximal subset satisfying

- $B \subset B'$,
- $B' \cap R = \emptyset$,
- $B' \cup R$ is $\varepsilon/8$ -discrete,
- B' is $\varepsilon/4$ -discrete.

(Note that $B' = B$ certainly satisfies these conditions.) Then by the maximality,

(\sharp) for every $y \in Y$, either $B(y, \varepsilon/8) \cap R \neq \emptyset$ or $B(y, \varepsilon/4) \cap B' \neq \emptyset$.

Then we have

Claim 1. *The set B' is $\varepsilon/2$ -dense.*

To show this, take $y \in Y$ with $B(y, \varepsilon/4) \cap B' = \emptyset$. Then by (\sharp), we can take a point $a \in B(y, \varepsilon/8) \cap R$. Since $\varepsilon < 1$ and each path-component of Y has the diameter ≥ 1 (see (\sharp 1) at the beginning of §3.1), there is a point $b \in Y$ such that $d(a, b) = \varepsilon/8$. Then $B(b, \varepsilon/8)$ does not meet R , because $a \in R$ and R is $\varepsilon/4$ -discrete. Again by (\sharp), it follows that $B(b, \varepsilon/4) \cap B' \neq \emptyset$. Fix an element $c \in B(b, \varepsilon/4) \cap B'$. Then we have

$$d(y, c) \leq d(y, a) + d(a, b) + d(b, c) < \varepsilon/8 + \varepsilon/8 + \varepsilon/4 = \varepsilon/2,$$

which proves the claim.

Let $R' \subset Y$ be a maximal subset satisfying

- $R \subset R'$,
- $B' \cap R' = \emptyset$,
- $B' \cup R'$ is $\varepsilon/8$ -discrete,
- R' is $\varepsilon/4$ -discrete.

(Note that $R' = R$ certainly satisfies these conditions.) Then by an argument similar to the above, we have

Claim 2. *The set R' is $\varepsilon/2$ -dense.*

Now the subsets $B', R' \subset Y$ satisfy $P_1(B', R', \varepsilon/2)$ and $P_2(B', R', \varepsilon/2)$. \square

3.2. Proof of the discrete approximation. Here by Γ we denote the set of all sequences of 0 and 1, that is, $\Gamma = \{0, 1\}^{\mathbb{N}}$.

We shall show that $\mathbf{F} = C(X, Y)$ satisfies the conditions (b) and (c) in Characterization Theorem 2.9. These conditions will be verified at the same time. In fact, we prove the following:

Proposition 3.5. *Let K_γ ($\gamma \in \Gamma$) be simplicial complexes. Then every map $\bigoplus_{\gamma \in \Gamma} f_\gamma: \bigoplus_{\gamma \in \Gamma} |K_\gamma| \rightarrow \mathbf{F}$ is approximable by a map $\bigoplus_{\gamma \in \Gamma} g_\gamma$ for which $\{g_\gamma(|K_\gamma|)\}_{\gamma \in \Gamma}$ is discrete in \mathbf{F} .*

This proposition clearly implies both of the conditions (b) and (c) for \mathbf{F} . To prove Proposition 3.5, we show the following:

Lemma 3.6 (main lemma). *There exists a modulus δ with the following property: for every continuous function $\alpha: \mathbf{F} \rightarrow (0, 1]$, every family $\{K_\gamma\}_{\gamma \in \Gamma}$ of simplicial complexes indexed by Γ and every map $\bigoplus_{\gamma \in \Gamma} f_\gamma: \bigoplus_{\gamma \in \Gamma} |K_\gamma| \rightarrow \mathbf{F}$, there exists a map $\bigoplus_{\gamma \in \Gamma} g_\gamma: \bigoplus_{\gamma \in \Gamma} |K_\gamma| \rightarrow \mathbf{F}$ such that*

- (i) $d(f_\gamma(p), g_\gamma(p)) < \alpha f_\gamma(p)$ for every $\gamma \in \Gamma$, $p \in |K_\gamma|$ and
- (ii) $d(g_\gamma(p), g_{\gamma'}(p')) \geq \min\{\delta \alpha f_\gamma(p), \delta \alpha f_{\gamma'}(p')\}$ for every $\gamma \neq \gamma' \in \Gamma$, $p \in |K_\gamma|$ and $p' \in |K_{\gamma'}|$.

In the above lemma, notice that there is no assumption on the simplicial complexes: they can possess arbitrarily many vertices and can be infinite-dimensional or non-locally finite. First we verify that Lemma 3.6 implies Proposition 3.5.

Proof of “Lemma 3.6 \Rightarrow Proposition 3.5”. It suffices to show that condition (ii) of Lemma 3.6 implies the discreteness of $\{g_\gamma(|K_\gamma|)\}$ in \mathbf{F} .

If $\{g_\gamma(|K_\gamma|)\}_{\gamma \in \Gamma}$ is not discrete, there are $F_0 \in \mathbf{F}$, a sequence $\{\gamma_j\}_{j \in \mathbb{N}}$ in Γ with $\gamma_j \neq \gamma_{j+1}$ ($j \in \mathbb{N}$), and $p_j \in |K_{\gamma_j}|$ ($j \in \mathbb{N}$) such that $g_{\gamma_j}(p_j) \rightarrow$

F_0 as $j \rightarrow \infty$. Let $f_j = f_{\gamma_j}$ and $g_j = g_{\gamma_j}$. Then by (ii), for each $j \in \mathbb{N}$ one of the following holds:

$$\begin{aligned} d(g_j(p_j), g_{j+1}(p_{j+1})) &\geq \delta \alpha f_j(p_j), \\ d(g_j(p_j), g_{j+1}(p_{j+1})) &\geq \delta \alpha f_{j+1}(p_{j+1}). \end{aligned}$$

Therefore, taking a suitable subsequence, we may assume that $\delta \alpha f_j(p_j) \rightarrow 0$, and hence $\alpha f_j(p_j) \rightarrow 0$ (notice that δ is a modulus). Now by (i) we have $d(f_j(p_j), g_j(p_j)) \rightarrow 0$, which means $f_j(p_j) \rightarrow F_0$. Then by the continuity of α , we see that $\alpha f_j(p_j) \rightarrow \alpha(F_0) > 0$. This contradicts $\alpha f_j(p_j) \rightarrow 0$. \square

For a simplicial complex K and its vertex $v \in K^{(0)}$, by $\text{st}(v, K)$ we mean the *closed star* at v in K , that is, the union of all (closed) simplexes which has v as a vertex. Recall the following classical lemma from J. H. C. Whitehead [18, Theorem 35]:

Lemma 3.7 (Whitehead). *For every simplicial complex K and open covering \mathcal{U} of $|K|$, there exists a subdivision K' of K such that the closed cover $\{\text{st}(v, K') ; v \in K'^{(0)}\}$ refines \mathcal{U} .* \square

Proof of Lemma 3.6. Fix a closed discrete subset

$$D = \left\{ x_j^{\beta, \rho} ; j \in \mathbb{N}, \beta, \rho \in \mathbb{N} \right\}$$

of X which is indexed by j, β, ρ . We also fix a discrete open collection $\{U_j^{\beta, \rho}\}_{j, \beta, \rho}$ with $x_j^{\beta, \rho} \in U_j^{\beta, \rho}$.

We take $B(i), R(i) \subset Y$ ($i \in \mathbb{N}$) as in Proposition 3.4. Namely, take $B(i)$ and $R(i)$ such that

- (1) $B(1) \subset B(2) \subset \cdots \subset Y$, $R(1) \subset R(2) \subset \cdots \subset Y$,
- (2) $B(i) \cap R(i) = \emptyset$,
- (3) $B(i) \cup R(i)$ is $2^{-(i+2)}$ -discrete,
- (4) Both $B(i)$ and $R(i)$ are 2^{-i} -dense.

Fix a modulus δ_0 as in Lemma 3.1. In particular, δ_0 satisfies the following:

- (5) $\|r(x) - r(x')\| \leq \varepsilon/2$ if $x, x' \in N$ and $\|x - x'\| < \delta_0(\varepsilon)$,
- (6) $\delta_0(\eta) < 1$ for each $\eta > 0$,
- (7) δ_0 is continuous and strictly monotone.

Then we define a modulus δ by

$$(8) \quad \delta = \frac{1}{32} \delta_0^2,$$

where $\delta_0^2 = \delta_0 \circ \delta_0$.

Let $\alpha: \mathbf{F} \rightarrow (0, 1]$ be a continuous function, $\{K_\gamma\}_{\gamma \in \Gamma}$ a family of simplicial complexes, and $\bigoplus_{\gamma \in \Gamma} f_\gamma: \bigoplus_{\gamma \in \Gamma} |K_\gamma| \rightarrow \mathbf{F}$ a map. We shall

show that there exists a map $\bigoplus_{\gamma \in \Gamma} g_\gamma: \bigoplus_{\gamma \in \Gamma} |K_\gamma| \rightarrow \mathbf{F}$ satisfying (i) and (ii) in the statement of Lemma 3.6.

To simplify the notation, we let

$$\alpha_\gamma = \delta_0^2 \alpha f_\gamma: |K_\gamma| \rightarrow (0, \infty) \quad (\gamma \in \Gamma).$$

We claim that there exists a subdivision K'_γ of K_γ such that

- (9) K'_γ has the form of the second barycentric subdivision, that is, there exists L_γ such that $K'_\gamma = \text{Sd}^2 L_\gamma$ ($\gamma \in \Gamma$),

and further that, writing $C_\gamma^v = \text{st}(v, L_\gamma)$ for $\gamma \in \Gamma$ and $v \in L_\gamma^{(0)}$,

$$(10) \quad p, q \in C_\gamma^v \text{ implies } \alpha_\gamma(p) \geq \frac{\alpha_\gamma(q)}{2},$$

$$(11) \quad \text{diam } f_\gamma(C_\gamma^v) \leq \frac{1}{2} \inf_{p \in C_\gamma^v} \alpha_\gamma(p).$$

To show this claim, take any $\gamma \in \Gamma$. Let \mathcal{U} be the open covering of $|K_\gamma|$ defined by

$$\mathcal{U} = \left\{ \alpha_\gamma^{-1}((t, 2t)) \cap f_\gamma^{-1} \left(B \left(F, \frac{\delta_0^2 \alpha(F)}{8} \right) \right) ; t > 0, F \in (\delta_0^2 \alpha)^{-1}((t, 2t)) \right\},$$

where $B \left(F, \frac{\delta_0^2 \alpha(F)}{8} \right)$ stands for an open ball in \mathbf{F} . It is easy to see that the above definition certainly gives a covering of $|K_\gamma|$. By using Lemma 3.7, we have a subdivision L_γ of K_γ such that

$$(12) \quad \{C_\gamma^v; v \in L_\gamma^{(0)}\} < \mathcal{U},$$

where $C_\gamma^v = \text{st}(v, L_\gamma)$. We shall prove that $K'_\gamma = \text{Sd}^2 L_\gamma$ satisfies our requirements (9)–(11). Clearly, (9) is satisfied. To see (10), let $v \in L_\gamma^{(0)}$ and $p, q \in C_\gamma^v$. Then by (12) there exists $t > 0$ such that $\alpha_\gamma(p), \alpha_\gamma(q) \in (t, 2t)$. Then $\alpha_\gamma(q)/\alpha_\gamma(p) < 2t/t = 2$, which means $\alpha_\gamma(q) < 2\alpha_\gamma(p)$.

To see (11), let $v \in L_\gamma^{(0)}$ and $p, q, r \in C_\gamma^v$. Then by (12) there exist $t > 0$ and $F \in \mathbf{F}$ such that

$$(13) \quad \alpha_\gamma(r), \delta_0^2 \alpha(F) \in (t, 2t) \text{ and}$$

$$(14) \quad d(f_\gamma(p), F), d(f_\gamma(q), F) < \frac{\delta_0^2 \alpha(F)}{8}.$$

It follows from (13) and (14) that

$$d(f_\gamma(p), f_\gamma(q)) < \frac{\delta_0^2 \alpha(F)}{4} < \frac{t}{2} < \frac{\alpha_\gamma(r)}{2}.$$

Since $p, q, r \in C_\gamma^v$ are arbitrary, we have (11).

Hereafter we write K_γ for K'_γ for simplicity. Let $L'_\gamma = \text{Sd } L_\gamma$ (which means $\text{Sd } L'_\gamma = K_\gamma$). Since the vertices of L'_γ are precisely the barycenters of simplexes in L_γ , the set $(L'_\gamma)^{(0)}$ of all vertices of L'_γ is represented as a disjoint union

$$(15) \quad (L'_\gamma)^{(0)} = \bigcup_{\theta \in \mathbb{N} \cup \{0\}} W_\gamma^\theta,$$

where W_γ^θ is the set of all barycenters of θ -simplexes in L_γ . We set $D_\gamma^w = \text{st}(w, K_\gamma)$ for $w \in (L'_\gamma)^{(0)} = \bigcup_{\theta=0}^\infty W_\gamma^\theta$.

Now define a metric $d = d_\gamma$ on the set $|K_\gamma|$ by

$$(16) \quad d(p, q) = d_\gamma(p, q) = \sum_{v \in K_\gamma^{(0)}} |p(v) - q(v)| \quad (p, q \in |K_\gamma|),$$

where $p(v)$ denotes the barycentric coordinate of p at v . This metric gives a topology of $|K_\gamma|$ not stronger than the original Whitehead weak topology. We note that

$$(17) \quad \text{if } w \text{ and } w' \text{ are distinct points in } W_\gamma^\theta \text{ for some } \theta \in \mathbb{N} \cup \{0\}, \text{ then } d(D_\gamma^w, D_\gamma^{w'}) \geq 2.$$

As a consequence, $\{D_\gamma^w\}_{w \in W_\gamma^\theta}$ is a discrete family of closed subsets in $|K_\gamma|$ for each $\theta \in \mathbb{N} \cup \{0\}$. By $D_\gamma(\theta)$ we denote the union of this family, that is, $D_\gamma(\theta) = \bigcup_{w \in W_\gamma^\theta} D_\gamma^w$. In view of (15), we observe that

$$(18) \quad |K_\gamma| = \bigcup_{w \in (L'_\gamma)^{(0)}} D_\gamma^w = \bigcup_{\theta=0}^\infty D_\gamma(\theta),$$

which is known as the dual cell decomposition of L_γ when L_γ is a combinatorial manifold.

As regards the relation between D_γ^w ($w \in (L'_\gamma)^{(0)}$) and C_γ^v ($v \in L_\gamma^{(0)}$), we easily see the following:

If there exists a simplex σ in L_γ such that v is a vertex of σ and w is the barycenter of σ , then $\overline{N}(D_\gamma^w, 1/2)$ is contained in C_γ^v .

From this we have

$$(19) \quad \{\overline{N}(D_\gamma^w, 1/2); w \in (L'_\gamma)^{(0)}\} < \{C_\gamma^v; v \in L_\gamma^{(0)}\}.$$

Now for each $v \in L_\gamma^{(0)}$, we let

$$(20) \quad \varepsilon_\gamma^v = \inf_{p \in C_\gamma^v} \alpha f_\gamma(p) \text{ for } v \in L_\gamma^{(0)},$$

which is positive by virtue of (10). Then define $\mu_\gamma^v \in \mathbb{N}$ by

$$\mu_\gamma^v = \min\{\mu \in \mathbb{N}; 2^{-\mu} < \delta_0^2(\varepsilon_\gamma^v)/2\}.$$

Notice that the modulus δ_0 is continuous. It is easy to see the following:

$$(21) \quad \text{diam } f_\gamma(C_\gamma^v) \leq \frac{\delta_0^2(\varepsilon_\gamma^v)}{2},$$

$$(22) \quad 2^{-\mu_\gamma^v} < \frac{\delta_0^2(\varepsilon_\gamma^v)}{2} \leq 2^{-(\mu_\gamma^v-1)}.$$

For each $\theta \in \mathbb{N} \cup \{0\}$, we define a function $\varphi_\gamma^\theta: |K_\gamma| \rightarrow [0, 1]$ by

$$(23) \quad \varphi_\gamma^\theta(p) = \max\{0, 1 - 2d(p, D_\gamma(\theta))\}.$$

We notice the following:

$$(24) \quad \varphi_\gamma^\theta|_{D_\gamma(\theta)} = 1, \quad \varphi_\gamma^\theta|_{|K_\gamma| \setminus N(D_\gamma(\theta), 1/2)} = 0,$$

$$(25) \quad \varphi_\gamma^\theta \text{ is continuous with respect to metric } d, \text{ hence to the Whitehead topology.}$$

For each $\gamma \in \Gamma$, $j \in \mathbb{N}$ and $\beta, \rho \in \mathbb{N}$, we define a map $h_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow E$. We fix γ, j, β and ρ . For brevity, we write $\check{f}^0 = \check{f}_\gamma(x_j^{\beta,\rho}, \cdot): |K_\gamma| \rightarrow Y$.

Case 1. If $\gamma(j) = 0$, we proceed as follows. In this case, we do not use ρ but we use β throughout the construction of $h_{\gamma,j}^{\beta,\rho}$. Take any $w \in W_\gamma^\beta \subset (L'_\gamma)^{(0)}$. By (19), we have $v = v(w) \in L_\gamma^{(0)}$ such that

$$(26) \quad \overline{N}(D_\gamma^w, 1/2) \subset C_\gamma^v.$$

Then by (4), we can choose $b_\gamma^w \in B(\mu_\gamma^v)$ satisfying

$$(27) \quad d(b_\gamma^w, \check{f}^0(C_\gamma^v)) < 2^{-\mu_\gamma^v}.$$

Now, for every $p \in \overline{N}(D_\gamma^w, 1/2)$, using (27), (26), (21), (22) and (20), we have

$$\begin{aligned} \|b_\gamma^w - \check{f}^0(p)\| &\leq d(b_\gamma^w, \check{f}^0(C_\gamma^v)) + \text{diam } \check{f}^0(C_\gamma^v) \\ &< 2^{-\mu_\gamma^v} + \frac{\delta_0^2(\varepsilon_\gamma^v)}{2} \\ &< \frac{\delta_0^2(\varepsilon_\gamma^v)}{2} + \frac{\delta_0^2(\varepsilon_\gamma^v)}{2} = \delta_0^2(\varepsilon_\gamma^v) \\ &\leq \delta_0^2 \alpha f_\gamma(p), \end{aligned}$$

which means that

$$(28) \quad \|b_\gamma^w - \check{f}^0(p)\| < \alpha_\gamma(p) \text{ for } w \in W_\gamma^\beta \text{ and } p \in \overline{N}(D_\gamma^w, 1/2).$$

Notice that, by (17),

$$(29) \quad \text{if } w \neq w' \in W_\gamma^\beta \text{ then } d(\overline{N}(D_\gamma^w, 1/2), \overline{N}(D_\gamma^{w'}, 1/2)) \geq 1.$$

We define $h_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow E$ by

$$(30) \quad h_{\gamma,j}^{\beta,\rho}(p) = \begin{cases} \varphi_\gamma^\beta(p)(b_\gamma^w - \check{f}^0(p)) & \text{if } p \in \overline{N}(D_\gamma^w, 1/2) \\ & \text{for (unique) } w \in W_\gamma^\beta, \\ 0 & \text{if } p \in |K_\gamma| \setminus N(D_\gamma(\beta), 1/2). \end{cases}$$

By (29) and (25), we see that $h_{\gamma,j}^{\beta,\rho}$ is continuous with respect to the Whitehead topology. For $w \in W_\gamma^\beta$ and $p \in \overline{N}(D_\gamma^w, 1/2)$, we have

$$(31) \quad \|h_{\gamma,j}^{\beta,\rho}(p)\| < \alpha_\gamma(p)$$

by (28). The same inequality is clearly true for $p \in |K_\gamma| \setminus \overline{N}(D_\gamma(\beta), 1/2)$. If we define $\check{f}^0 + h_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow E$ by the addition in E , we have

$$(32) \quad (\check{f}^0 + h_{\gamma,j}^{\beta,\rho})(D_\gamma^w) = \{b_\gamma^w\} \subset B(\mu_\gamma^{v(w)}).$$

by (30) and (24).

Case 2. If $\gamma(j) = 1$, we do not use β but we use ρ . For each $w \in W_\gamma^\rho$, we can take $v = v(w) \in L_\gamma^{(0)}$ satisfying $\overline{N}(D_\gamma^w, 1/2) \subset C_\gamma^v$ by (19). By (4), we can choose $r_\gamma^w \in R(\mu_\gamma^{v(w)})$ with $d(r_\gamma^w, \check{f}^0(C_\gamma^v)) < 2^{-\mu_\gamma^v}$. Then we can prove

$$(33) \quad \|r_\gamma^w - \check{f}^0(p)\| < \alpha_\gamma(p)$$

in the same way as (28) in Case 1. We define a map $h_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow E$ by

$$(34) \quad h_{\gamma,j}^{\beta,\rho}(p) = \begin{cases} \varphi_\gamma^\rho(p)(r_\gamma^w - \check{f}^0(p)) & \text{if } p \in \overline{N}(D_\gamma^w, 1/2) \\ & \text{for (unique) } w \in W_\gamma^\rho, \\ 0 & \text{if } p \in |K_\gamma| \setminus N(D_\gamma(\rho), 1/2). \end{cases}$$

Then by (33), for $p \in |K_\gamma|$,

$$(35) \quad \|h_{\gamma,j}^{\beta,\rho}(p)\| < \alpha_\gamma(p),$$

which is the same inequality as (31). If we define $\check{f}^0 + h_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow E$ using the addition in E ,

$$(36) \quad (\check{f}^0 + h_{\gamma,j}^{\beta,\rho})(D_\gamma^w) = \{r_\gamma^w\} \subset R(\mu_\gamma^{v(w)}) \text{ for every } w \in W_\gamma^\rho.$$

We have defined $h_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow E$ for every $\gamma \in \Gamma, j \in \mathbb{N}$ and $\beta, \rho \in \mathbb{N} \cup \{0\}$. Now we claim the following.

Claim 3. $\{h_{\gamma,j}^{\beta,\rho}\}_{j,\beta,\rho}$ is equicontinuous for every $\gamma \in \Gamma$.

First we fix $\gamma \in \Gamma$. We show the equicontinuity of

$$\{h_{\gamma,j}^{\beta,\rho}\}_{j \in \gamma^{-1}(0), \beta, \rho \in \mathbb{N}}$$

and that of

$$\{h_{\gamma,j}^{\beta,\rho}\}_{j \in \gamma^{-1}(1), \beta, \rho \in \mathbb{N}}$$

separately. Since the latter family is treated similarly, we prove only the equicontinuity of the former. Take any point $p \in |K_\gamma|$. It suffices to show the equicontinuity at the point p with respect to the metric d

defined in (16). In view of the definition (30), it suffices to consider the case $d(p, D_\gamma(\beta)) \leq 1/2$ only, since otherwise $h_{\gamma,j}^{\beta,\rho}$'s, where $j \in \gamma^{-1}(0)$, are constantly zero in a common neighborhood of p . Then there exists (unique) $w \in W_\gamma^\beta$ such that $d(p, D_\gamma^w) \leq 1/2$.

Let $p' \in |K_\gamma|$ be any point satisfying $d(p', p) < 1/2$ and let $j \in \mathbb{N}$, $\beta, \rho \in \mathbb{N} \cup \{0\}$ and $\check{f}^0 = \check{f}_\gamma(x_j^{\beta,\rho}, \cdot): |K_\gamma| \rightarrow Y$. We distinguish two cases:

Case A. If $p' \in |K| \setminus N(D_\gamma(\beta), 1/2)$, then $h_{\gamma,j}^{\beta,\rho}(p') = 0$ by the definition (30). Then by (28) and (23), we have

$$\begin{aligned} \|h_{\gamma,j}^{\beta,\rho}(p') - h_{\gamma,j}^{\beta,\rho}(p)\| &= \|h_{\gamma,j}^{\beta,\rho}(p)\| \\ &= \varphi_\gamma^\beta(p) \|b_\gamma^w - \check{f}^0(p)\| \\ &\leq \varphi_\gamma^\beta(p) \alpha_\gamma(p) \\ &= (1 - 2d(p, D_\gamma^w)) \alpha_\gamma(p). \end{aligned}$$

Now, making use of the assumption that $d(p', D_\gamma^w) \geq 1/2$, we have $d(p, p') \geq 1/2 - d(p, D_\gamma^w)$, which is also expressed as $1 - 2d(p, D_\gamma^w) \leq 2d(p, p')$. Combining this with the above inequality, we have

$$\|h_{\gamma,j}^{\beta,\rho}(p') - h_{\gamma,j}^{\beta,\rho}(p)\| \leq 2d(p, p') \alpha_\gamma(p) = 2d(p, p') \delta_0^2 \alpha f_\gamma(p).$$

Therefore by (6), we have

$$(37) \quad \|h_{\gamma,j}^{\beta,\rho}(p') - h_{\gamma,j}^{\beta,\rho}(p)\| \leq 2d(p, p').$$

Case B. If $p' \in \overline{N}(D_\gamma(\beta), 1/2)$, we must have $p' \in \overline{N}(D_\gamma^w)$ by (17), since $d(p', p) < 1/2$ and $d(p, D_\gamma^w) < 1/2$. Consider $v(w) \in L_\gamma^{(0)}$ and $b_\gamma^w \in B(\mu_\gamma^{v(w)})$ that are chosen when we defined $h_{\gamma,j}^{\beta,\rho}$. Then we have

$$\begin{aligned} \|h_{\gamma,j}^{\beta,\rho}(p') - h_{\gamma,j}^{\beta,\rho}(p)\| &= \|\varphi_\gamma^\beta(p)(b_\gamma^w - \check{f}^0(p)) - \varphi_\gamma^\beta(p')(b_\gamma^w - \check{f}^0(p'))\| \\ &\leq |\varphi_\gamma^\beta(p) - \varphi_\gamma^\beta(p')| \|b_\gamma^w - \check{f}^0(p)\| \\ &\quad + |\varphi_\gamma^\beta(p')| \|\check{f}^0(p) - \check{f}^0(p')\| \\ &= 2|d(p, D_\gamma^w) - d(p', D_\gamma^w)| \|b_\gamma^w - \check{f}^0(p)\| \\ &\quad + |\varphi_\gamma^\beta(p')| \|\check{f}^0(p) - \check{f}^0(p')\| \\ &\leq 2d(p, p') \|b_\gamma^w - \check{f}^0(p)\| + d(f_\gamma(p), f_\gamma(p')) \\ &\leq 2d(p, p') \alpha_\gamma(p) + d(f_\gamma(p), f_\gamma(p')), \end{aligned}$$

where the last inequality follows from (28). By (6), we have

$$(38) \quad \|h_{\gamma,j}^{\beta,\rho}(p') - h_{\gamma,j}^{\beta,\rho}(p)\| \leq 2d(p, p') + d(f_\gamma(p), f_\gamma(p')).$$

The inequalities (37) and (38) shows that the family $\{h_{\gamma,j}^{\beta,\rho}\}_{j \in \gamma^{-1}(0), \beta, \rho \in \mathbb{N}}$ is equicontinuous at $p \in |K_\gamma|$. The proof of Claim 3 is complete.

We define $g_{\gamma,j}^{\beta,\rho}: |K_\gamma| \rightarrow Y$ ($\gamma \in \Gamma, j \in \mathbb{N}, \beta, \rho \in \mathbb{N}$) by

$$(39) \quad g_{\gamma,j}^{\beta,\rho}(p) = r(\check{f}^0(p) + h_{\gamma,j}^{\beta,\rho}(p)),$$

where $\check{f}^0 = \check{f}_\gamma(x_j^{\beta,\rho}, \cdot): |K_\gamma| \rightarrow Y$ and the addition is performed in E . By (35) and (6) we have $\|h_{\gamma,j}^{\beta,\rho}(p)\| < \alpha_\gamma(p) < 1$, hence

$$\check{f}^0(p) + h_{\gamma,j}^{\beta,\rho}(p) \in N(Y, 1) \quad (p \in |K_\gamma|),$$

which shows that $g_{\gamma,j}^{\beta,\rho}$ is well-defined. Since r is a uniformly continuous retraction onto Y , we conclude from Claim 3 that

$$(40) \quad \{g_{\gamma,j}^{\beta,\rho}\}_{j,\beta,\rho} \text{ is equicontinuous for every } \gamma \in \Gamma.$$

Notice that $\check{f}_\gamma(x_j^{\beta,\rho}, p) = \check{f}^0(p) = r(\check{f}^0(p))$ ($p \in |K_\gamma|$). Then by (31), (35) and (5),

$$(41) \quad d(g_{\gamma,j}^{\beta,\rho}(p), \check{f}_\gamma(x_j^{\beta,\rho}, p)) < \delta_0 \alpha f_\gamma(p).$$

Furthermore, by (32), (36) and the definition (39) of $g_{\gamma,j}^{\beta,\rho}$, we have

$$(42) \quad g_{\gamma,j}^{\beta,\rho}(D_\gamma^w) = \{b_\gamma^w\} \subset B(\mu_\gamma^{v(w)}) \text{ if } \gamma(j) = 0 \text{ and } w \in W_\gamma^\beta,$$

$$(43) \quad g_{\gamma,j}^{\beta,\rho}(D_\gamma^w) = \{r_\gamma^w\} \subset R(\mu_\gamma^{v(w)}) \text{ if } \gamma(j) = 1 \text{ and } w \in W_\gamma^\rho.$$

By Lemma 3.2, (40) and (41), for each $\gamma \in \Gamma$ there exists a map $g_\gamma: |K_\gamma| \rightarrow \mathbf{F}$ satisfying

$$(44) \quad \check{g}_\gamma(x_j^{\beta,\rho}, \cdot) = g_{\gamma,j}^{\beta,\rho} \quad (j \in \mathbb{N}, \beta, \rho \in \mathbb{N}),$$

$$(45) \quad d(g_\gamma(p), f_\gamma(p)) < \alpha f_\gamma(p) \quad (p \in |K_\gamma|),$$

and $\check{g}_\gamma|_{(X \setminus U) \times |K_\gamma|} = \check{f}_\gamma|_{(X \setminus U) \times |K_\gamma|}$, where $U = \bigcup_{j,\beta,\rho} U_j^{\beta,\rho}$. The above inequality (45) means that the condition (i) is satisfied. In the rest of the proof, we show that the condition (ii) is also satisfied.

Let $\gamma \neq \gamma' \in \Gamma$, $p \in |K_\gamma|$ and $p' \in |K_{\gamma'}|$. We may assume that $\gamma(j_0) = 0$ and $\gamma'(j_0) = 1$ for some $j_0 \in \mathbb{N}$. By (18), there exist $\beta_0, \rho_0 \in \mathbb{N}$, $w_0 \in W_\gamma^{\beta_0}$ and $w'_0 \in W_{\gamma'}^{\rho_0}$ such that $p \in D_\gamma^{w_0}$ and $p' \in D_{\gamma'}^{w'_0}$.

Let $x_0 = x_{j_0}^{\beta_0, \rho_0}$, $g_0 = g_{\gamma, j_0}^{\beta_0, \rho_0}$ and $g'_0 = g_{\gamma', j_0}^{\beta_0, \rho_0}$. Moreover we let $D_0 = D_\gamma^{w_0}$ and $D'_0 = D_{\gamma'}^{w'_0}$. Then by (44) we have $\check{g}_\gamma(x_0, \cdot) = g_0$ and $\check{g}_{\gamma'}(x_0, \cdot) = g'_0$. Furthermore by (42) and (43)

$$(46) \quad g_0(p) = b_\gamma^{w_0} \in B(\mu_\gamma^{v(w_0)}),$$

since $\gamma(j_0) = 0$. Similarly, by (43),

$$(47) \quad g'_0(p') = r_{\gamma'}^{w'_0} \in R(\mu_{\gamma'}^{v(w'_0)}),$$

since $\gamma'(j_0) = 1$. We let $v_0 = v(w_0) \in L_\gamma^{(0)}$ and $v'_0 = v(w'_0) \in L_{\gamma'}^{(0)}$. Further, we let $\mu_0 = \mu_\gamma^{v_0}$ and $\mu'_0 = \mu_{\gamma'}^{v'_0}$. By (1), (2) and (3) we have

$$(48) \quad d(g_0(p), g'_0(p')) \geq d(B(\mu_0), R(\mu'_0)) \geq \frac{1}{4} \min\{2^{-\mu_0}, 2^{-\mu'_0}\}.$$

On the other hand, by (10), we have

$$(49) \quad \alpha_\gamma(q) \geq \alpha_\gamma(p)/2 \text{ for every } q \in C_\gamma^{v_0} \text{ and}$$

$$(50) \quad \alpha_{\gamma'}(q') \geq \alpha_{\gamma'}(p')/2 \text{ for every } q' \in C_{\gamma'}^{v'_0}.$$

Then, by using (46)–(50), (22), (20) as well as (7) (to exchange δ_0 and infimum), we have the following:

$$\begin{aligned} d(g_\gamma(p), g_{\gamma'}(p')) &\geq d(\check{g}_\gamma(x_0, p), \check{g}_{\gamma'}(x_0, p')) = d(g_0(p), g'_0(p')) \\ &\geq \frac{1}{4} \min\{2^{-\mu_0}, 2^{-\mu'_0}\} \\ &\geq \frac{1}{4} \cdot \frac{1}{4} \{\delta_0^2(\varepsilon_\gamma^{v_0}), \delta_0^2(\varepsilon_{\gamma'}^{v'_0})\} \\ &= \frac{1}{16} \min \left\{ \inf_{q \in D_0} \alpha_\gamma(q), \inf_{q' \in D'_0} \alpha_{\gamma'}(q') \right\} \\ &\geq \frac{1}{32} \min\{\alpha_\gamma(p), \alpha_{\gamma'}(p')\} \\ &= \frac{1}{32} \min\{\delta_0^2 \alpha f_\gamma(p), \delta_0^2 \alpha f_{\gamma'}(p')\} \end{aligned}$$

By (8), this inequality means

$$d(g_\gamma(p), g_{\gamma'}(p')) \geq \min\{\delta \alpha f_\gamma(p), \delta \alpha f_{\gamma'}(p')\}.$$

This show that the condition (ii) is satisfied. The proof is complete. \square

This lemma implies Proposition 3.5, as is seen before. Then it follows that the function space \mathbf{F} satisfies the conditions (b) and (c). We have verified that \mathbf{F} satisfies the conditions (a), (b), (c) in Theorem 2.9 and the proof of Theorem 1.1 is completed.

4. REMARKS ON THEOREM 1.1

4.1. The role of the condition on the components. In Theorem 1.1, the assumption that Y has the diameters of components away from zero is essential in the following sense:

Theorem 4.1. *Let (Y, d) be a separable complete ANRU. If $C(\mathbb{N}, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold, then the diameters of components of Y are bounded away from zero.*

Proof. First notice that Y has no isolated points. Indeed, if Y has a isolated point p , the constant map $\mathbb{N} \rightarrow Y$ with the value p is an isolated point in $C(\mathbb{N}, Y)$, which means that $C(\mathbb{N}, Y)$ is not an $\ell^2(2^{\aleph_0})$ -manifold, contradicting to the assumption.

Thus, to prove our conclusion, it is enough to consider the case where Y has infinitely many path-components C_i ($i \in \mathbb{N}$).

First we show that $\{d(C_i, C_j); i \neq j \in \mathbb{N}\}$ has a positive lower bound. Suppose this is not true. We can embed Y isometrically into a normed space E as a linearly independent closed set (Bessaga-Pełczyński [4, Chapter II, Corollary 1.1]). For $i < j$, take a line segment $L_{i,j}$ in E connecting a point in C_i and a point in C_j such that $\text{diam } L_{i,j} < 2d(C_i, C_j)$. Then the diameters of $L_{i,j}$'s do not have positive lower bound. Since Y is linearly independent in E , the segments $L_{i,j}$ are disjoint except for their endpoints: that is, if $p \in L_{i,j} \cap L_{k,l}$ for $(i, j) \neq (k, l)$, then p is a common endpoint of $L_{i,j}$ and $L_{k,l}$. Moreover, each $L_{i,j}$ intersects Y only in their endpoints.

Consider the subset $Z = Y \cup \bigcup_{i < j} L_{i,j}$ of E . Then Y is closed in Z . Since Y is an ANRU, there exist a uniform neighborhood N and a retraction $r: N \rightarrow Y$. Then for some i, j , the segment $L_{i,j}$ is contained in N . Then, the image $r(L_{i,j})$ is contained in Y and intersects both C_i and C_j . This means that $r(L_{i,j})$ is not path-connected, a contradiction.

Thus, the set $\{d(C_i, C_j); i \neq j \in \mathbb{N}\}$ has a lower bound $\delta > 0$. To obtain the conclusion, suppose that the diameters of components are not bounded away from zero. Then there exists a subsequence $\{C'_i\}_{i \in \mathbb{N}}$ of $\{C_i\}_{i \in \mathbb{N}}$ such that $\text{diam } C'_i \rightarrow 0$ as $i \rightarrow \infty$. Hereafter we simply write C_i to mean C'_i . For each $i \in \mathbb{N}$, fix a point $p_i \in C_i$ and a countable dense subset D_i of C_i . Consider an element $f_0 \in C(\mathbb{N}, Y)$ defined by $f_0(i) = p_i$ and a subset

$$U = \{f \in C(\mathbb{N}, Y); f(i) \in C_i \text{ for all } i \in \mathbb{N}\}$$

of $C(\mathbb{N}, Y)$. Then U is a neighborhood of f_0 in $C(\mathbb{N}, Y)$. Indeed, U contains the δ -neighborhood of f_0 . Moreover, U is separable. Indeed,

$$\bigcup_{j \in \mathbb{N}} \{f \in C(\mathbb{N}, Y); f(i) \in D_i \text{ if } i \leq j \text{ and } f(i) = p_i \text{ if } i > j\}$$

is a countable dense subset of U , since $\text{diam } C_i \rightarrow 0$ as $i \rightarrow \infty$. This means that $f_0 \in C(\mathbb{N}, Y)$ has a separable neighborhood, which contradicts the assumption that $C(\mathbb{N}, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold. \square

Thus, we have a refinement of Theorem 1.1 as follows:

Corollary 4.2. *Let (Y, d) be a separable complete ANRU. The following are equivalent:*

- (1) $C(X, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold for every separable noncompact metrizable space X .
- (2) $C(\mathbb{N}, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold.
- (3) The diameters of path-components of Y are bounded away from zero.

□

4.2. Remarks on the space of bounded maps and counterexamples. Consider the subspace $C_B(X, Y)$ of all bounded continuous maps from X to Y . Then $C_B(X, Y)$ is open in $C(X, Y)$, and hence $C_B(X, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold if $C(X, Y)$ is. The converse is false, and this will be shown in Example 4.4. The following will be used in the proof of Corollary 1.3:

Corollary 4.3. *Let X be a separable noncompact metrizable space and (Y, d) a metric space. Suppose that Y can be represented as $Y = \bigcup_{i \in \mathbb{N}} Y_i$, where $\{Y_i\}_{i \in \mathbb{N}}$ satisfies the following properties:*

- $\{Y_i\}_{i \in \mathbb{N}}$ is an increasing sequence of complete ANRU's,
- For each $i \in \mathbb{N}$, the diameters of the path-components of Y_i are bounded away from zero,
- For each $i \in \mathbb{N}$, we have either $d(Y_i, Y \setminus Y_{i+1}) > 0$ or $Y_i = Y$.
- Every bounded subset of Y is contained in some Y_i .

Then $C_B(X, Y)$ is an $\ell^2(2^{\aleph_0})$ -manifold.

Proof. We can think of $C_B(X, Y_i)$ as a subset of $C_B(X, Y)$. By the first two assumptions for $\{Y_i\}$ and Theorem 1.1, $C_B(X, Y_i)$ is an $\ell^2(2^{\aleph_0})$ -manifold for each $i \in \mathbb{N}$. The third assumption shows that $C_B(X, Y_{i+1})$ is a neighborhood of $C_B(X, Y_i)$ in $C_B(X, Y)$. The last assumption means $C_B(X, Y) = \bigcup_{i \in \mathbb{N}} C_B(X, Y_i)$. Thus, we have the conclusion. □

Example 4.4. Let $Y \subset \mathbb{R}^2$ be the non-ANRU homeomorphic to \mathbb{R} as in Example 2.8. If we let $Y_n = \{(x, y) \in Y; x \leq n\}$, then $\{Y_n\}_{n \in \mathbb{N}}$ satisfies the conditions in Corollary 4.3. Therefore for every separable noncompact metrizable space X , the space $C_B(X, Y)$ of bounded continuous functions is an $\ell^2(2^{\aleph_0})$ -manifold.

However, the space $C(X, Y)$ is not necessarily a $\ell^2(2^{\aleph_0})$ -manifold. Indeed, $C(\mathbb{N}, Y)$ is not locally connected at f_0 defined by $f_0(m) = (m, 1/m)$. To see this, take any $\varepsilon > 0$ and choose $n \in \mathbb{N}$ so that $2/n < \varepsilon$. Then define $f \in C(\mathbb{N}, Y)$ as follows: $f(n) = (n, -1/n)$ and $f|_{\mathbb{N} \setminus \{n\}} = f_0|_{\mathbb{N} \setminus \{n\}}$. Then, $d(f_0, f) = 2/n < \varepsilon$ but f_0 and f are not connected by a path with diameter < 1 . □

The compact-open topology is another important topology on $C(X, Y)$. We write $C_k(X, Y)$ to denote $C(X, Y)$ with this topology, while $C(X, Y)$ means the space with the uniform topology as before.

Example 4.5. The space $C_k(X, Y)$ can fail to be locally path-connected (and hence to be a Hilbert manifold), even if $C(X, Y)$ is a Hilbert manifold.

For example, let

$$X = Y = \bigcup_{n \in \mathbb{Z}} \{(x, y) \in \mathbb{R}^2; (x - 2n)^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

Then, Y can be regarded as a metric space as a subset of \mathbb{R}^2 and is uniformly equivalent to a 1-dimensional uniform complex (see Proposition 2.6). This means that Y is an ANRU by Corollary 2.5. Thus, $C(X, Y)$ is a $\ell^2(2^{\aleph_0})$ -manifold by Theorem 1.1.

We shall prove that $C_k(X, Y)$ is not locally path-connected at the identity map $X \rightarrow Y$. For every neighborhood V of the identity in $C_k(X, Y)$, there exists $n \in \mathbb{N}$ such that the map $c_n: X \rightarrow Y$ belongs to V , where c_n is defined by

$$c_n(x, y) = \begin{cases} (x, y) & \text{if } |x| \leq n, \\ (n, 0) & \text{if } x \geq n, \\ (-n, 0) & \text{if } x \leq -n. \end{cases}$$

Then c_n is not homotopic to the identity, that is, there is no path in $C_k(X, Y)$ from c_n to the identity. \square

Example 4.6. The space $C(X, Y)$ can fail to be locally path-connected even if $C_k(X, Y)$ is homeomorphic to the Hilbert space. Let X, Y be as in Example 4.4. Then, $C_k(X, Y) \approx C_k(\mathbb{N}, \mathbb{R}) \approx \mathbb{R}^{\mathbb{N}} \approx \ell^2$, where the last homeomorphism is proved in Anderson-Bing [1], whereas $C(X, Y)$ is not locally path-connected as seen before. \square

A popular notion similar to ANRU is that of *uniform ANR* introduced by Michael [11]. Every ANRU is clearly a uniform ANR. We do not know if our assumption that “ Y is an ANRU” can be loosened to “ Y is a uniform ANR”, so we ask the next question:

Question 4.7. Let X be a separable, noncompact metric space and let (Y, d) be a separable, path-connected complete metric space that is a uniform ANR. Then, is $C(X, Y)$ an ANR? Is it an $\ell^2(2^{\aleph_0})$ -manifold?

5. BOUNDED MAPS INTO COMPLETE RIEMANNIAN MANIFOLDS

In this section we apply Corollary 4.3 to the case where Y is a connected complete Riemannian manifold (of class C^∞) to prove Corollary

1.3. Let K be a nonempty compact subset of M . We say that K is *strictly convex* if every two distinct points $p, q \in K$ can be joined by a unique geodesic contained in K and this geodesic is contained in the interior $\text{Int } K$ except for its endpoints. Clearly if two strictly convex sets meet, then the intersection is strictly convex. The next lemma is known in the folklore and is easy to prove:

Lemma 5.1. *Let M be an n -dimensional Riemannian manifold without boundary. If $V \subset M$ is strictly convex and has nonempty interior in M , then V is homeomorphic to I^n . \square*

We use the following gluing theorem for ANRU's:

Theorem 5.2 (Nguyen To Nhu [13]). *Let (X, d) be a metric space and A_1, A_2 closed subspaces with $A_1 \cap A_2 \neq \emptyset$ and $A_1 \cup A_2 = X$. Then (X, d) is an ANRU if the following (i) and (ii) hold:*

- i) A_1, A_2 and $A_1 \cap A_2$ are ANRU's,
- ii) the metric d on X is uniformly equivalent to d' , where d' is defined by

$$d'(x, y) = \begin{cases} d(x, y) & x, y \in A_i \ (i = 1, 2), \\ \inf\{d(x, z) + d(y, z) ; z \in A_1 \cap A_2\} & \text{otherwise.} \end{cases}$$

Notice that if X is compact, the condition (ii) for d' is always satisfied. In this case, we can even drop the condition $A_1 \cap A_2 \neq \emptyset$.

Proof of Corollary 1.3. Let M be a connected complete Riemannian manifold. It suffices to show that M satisfies the assumption for Y in Corollary 4.3. We can cover M with locally finite collection of countably many strictly convex sets V_i ($i \in \mathbb{N}$) such that the interiors $U_i = \text{Int } V_i$ are nonempty. Indeed, we can take such V_i 's to be closed balls $V_i = \bar{B}(p_i, r_i)$, with the interior $U_i = \text{Int } V_i$ being the open ball $B(p_i, r_i)$ (see Lang [10, Chapter VIII, Theorem 5.8 and Lemma 5.9]). Further we assume that $V_{i_1} \cap \cdots \cap V_{i_n} \neq \emptyset$ implies $U_{i_1} \cap \cdots \cap U_{i_n} \neq \emptyset$ for any finitely many i_1, \dots, i_n . This is realized by shrinking each V_i slightly.

Let $M_i = \bigcup_{k=1}^i V_k$ ($i \in \mathbb{N}$). We show that every M_i is an ANRU by induction on i . For $i = 1$, $M_1 = V_1$ is strictly convex set with nonempty interior and hence is homeomorphic to I^n by Lemma 5.1. Thus V_1 is an ANRU by Proposition 2.6. For $i = 2$, V_1 and V_2 are ANRU's by the case $i = 1$. If V_1 does not meet V_2 , then $M_2 = V_1 \cup V_2$ is an ANRU as noticed above. If V_1 meets V_2 , then U_1 also meets U_2 , and hence $V_1 \cap V_2$ is again a strictly convex set with nonempty interior, which is an ANRU. By Theorem 5.2, $M_1 = V_1 \cup V_2$ is an ANRU. Suppose that

$i \geq 3$ and the assertion is true for $i - 1$. Then $M_i = M_{i-1} \cup V_i$ and

$$M_{i-1} \cap V_i = \bigcup_{k=1}^{i-1} (V_k \cap V_i).$$

In the above, the right hand side is an ANRU by the inductive hypothesis. Then it follows that M_j is an ANRU by Theorem 5.2.

We can replace $\{M_i\}$ with a suitable subsequence such that $M_i \subset \text{Int } M_{i+1}$ for each i . Then, $\{M_i\}$ satisfies the conditions for $\{Y_i\}_{i \in \mathbb{N}}$ in Corollary 4.3. Indeed, the last condition holds since any bounded closed set in a complete Riemannian manifold is compact (see Lang [10, Chapter VIII, Corollary 6.7]). \square

Example 5.3. In Corollary 1.3, we cannot replace $C_B(X, Y)$ by $C(X, Y)$. We can modify the embedding $i: \mathbb{R} \rightarrow \mathbb{R}^2$ given in Example 2.8 slightly to get a C^∞ -embedding $i': \mathbb{R} \rightarrow \mathbb{R}^2$ which agrees with i outside small neighborhoods of 1 and -1 . Then, the image $Y = i'(\mathbb{R})$ is naturally a complete connected Riemannian manifold. The same argument as in Example 4.4 shows that the space $C(\mathbb{N}, Y)$ is not an $\ell^2(2^{\aleph_0})$ -manifold. \square

6. GENERALIZATION OF THE DOMAIN SPACE

Neither the separability nor the metrizability of the domain space X is essential,³ but still some restrictions corresponding to noncompactness of X , which is necessary to take a closed discrete set of “sufficiently many” points in X , should be made. We can also loosen the condition on the density of Y without essential change.

Recall that the *density* of a space is the minimal cardinality of dense sets and that the *extent* of a space is the supremum of the cardinalities of closed discrete sets. For metrizable spaces they both coincide with the weight of the space. We say that a space *attains* its extent if there exists a closed discrete subset whose cardinality equals to the extent. Note that a separable noncompact metric space attains its extent \aleph_0 .

Lemma 6.1. *Let X be a normal space whose density λ is equal to the extent and Y be a metrizable space of weight ν which has a nontrivial path-component. If X attains its extent and $\nu \leq \lambda$, then the weight of $C_B(X, Y)$ is equal to 2^λ .*

Proof. Analogous to Lemma 2.10 (i). Notice that if $\nu \leq \lambda$ then $2^\lambda \leq \nu^\lambda \leq (2^\nu)^\lambda = 2^{\nu\lambda} = 2^\lambda$. \square

By this lemma, Theorem 1.1 is generalized as follows:

³This remark is due to D. Gauld.

Theorem 6.2. *Let X be a normal space whose density λ is equal to the extent and (Y, d) an complete ANRU of weight not greater than λ . If X attains its extent and the diameters of path-components of Y are bounded away from zero, then $C_B(X, Y)$ is an $\ell^2(2^\lambda)$ -manifold. \square*

If the domain space X is metrizable, the property that X attains its extent is described by a covering property similar to compactness. According to Barbati and Costantini [2], a metrizable space Z is *generalized compact* (abbreviated GK) if every open cover of Z has a subcover of cardinality (strictly) less than its weight. Costantini [5, Proposition 2.3] has proved that a metrizable space attains its extent if and only if it is not GK. Hence, we have the following:

Theorem 6.3. *Let X be a metrizable, not GK space of weight λ and (Y, d) an complete ANRU of weight not greater than λ . If the diameters of path-components of Y are bounded away from zero, then $C_B(X, Y)$ is an $\ell^2(2^\lambda)$ -manifold. \square*

APPENDIX. THE EQUIVALENCE OF ANRU AND ANEU

In this appendix, we prove Proposition 2.4 which states the equivalence of ANRU and ANEU.⁴ The “if” part of the proposition is easy. To prove the “only if” part, it suffices to show the following:

Lemma A.1. *Let X, Y be metric spaces and $A \subset Y$ a nonempty closed set. For every uniformly continuous map $f: A \rightarrow X$, there exist an isometric closed embedding $X \hookrightarrow Z$ into a metric space Z and a uniformly continuous extension $\tilde{f}: N(A, \varepsilon) \rightarrow Z$ for some $\varepsilon > 0$.*

The next lemma is a special case when the attaching map is an isometry.

Lemma A.2 (Nguyen To Nhu [13]). *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $A \subset X$ a nonempty closed set. Let $f: A \rightarrow Y$ be an isometric embedding. Consider the disjoint union $X \sqcup Y$ as a set and the equivalence relation \sim generated by $a \sim f(a), a \in A$. Then we can define a metric d on the set $Z = X \sqcup Y / \sim$ as follows:*

$$d(x, y) = \begin{cases} d_X(x, y) & x, y \in X, \\ d_Y(x, y) & x, y \in Y, \\ \inf_{a \in A} (d_X(x, a) + d_Y(f(a), y)) & x \in X, y \in Y. \end{cases}$$

Furthermore, Y is naturally isometrically embedded as a closed set in Z and f naturally extends to an isometric embedding $\tilde{f}: X \rightarrow Z$. \square

⁴The following argument is essentially due to Nguyen To Nhu [12]. The author is also indebted to K. Mine for useful suggestions for this appendix.

We prepare the results required to show Lemma A.1. First, we need a basic result on ANEU's, which is proved by Isbell [8, Theorem 1.4].⁵

Lemma A.3. *The real line \mathbb{R} is an ANEU with respect to the usual metric.* \square

Let D be a set and Y a metric space. Define $F(D, Y)$ to be the space of all functions from D into Y endowed with a metric $d(f, g) = \min\{1, \sup_{x \in D} d(f(x), g(x))\}$. We think of D as a metric space by the standard discrete metric. Since a function $f: Z \rightarrow F(D, Y)$ from a metric space Z is uniformly continuous if and only if the corresponding function $\check{f}: D \times Z \rightarrow Y$ is uniformly continuous, we can easily show the following:

Lemma A.4. *If Y is an ANEU, then $F(D, Y)$ is an ANEU. In particular, $F(D, \mathbb{R})$ is an ANEU.* \square

This lemma enables us to prove:

Proposition A.5. *Let (X, d) be a metric space and ρ a uniformly continuous pseudometric on a closed subset $A \subset X$. Then, for some $\varepsilon > 0$, the pseudometric ρ can be extended to a uniformly continuous pseudometric $\tilde{\rho}$ on $N = N(A, \varepsilon)$ such that A is closed in $(N, \tilde{\rho})$.*

Proof. Define $g: A \rightarrow F(A, \mathbb{R})$ by $g(x)(y) = \rho(x, y)$. Since ρ is uniformly continuous, g is uniformly continuous. By Lemma A.4, there exist $\varepsilon > 0$ and a uniformly continuous extension $\tilde{g}: N = N(A, \varepsilon) \rightarrow F(A, \mathbb{R})$. Notice that, for $x, y \in A$,

$$\rho(x, y) = \sup_{a \in A} |\tilde{g}(x)(a) - \tilde{g}(y)(a)|.$$

Define $\tilde{\rho}_0: N \times N \rightarrow \mathbb{R}$ by the same formula above (this is well-defined for sufficiently small ε). By definition $\tilde{\rho}_0$ is an extension of ρ and a uniformly continuous pseudometric on N . With respect to this pseudometric, the subset A may not be closed. We can define the desired pseudometric $\tilde{\rho}$ by $\tilde{\rho}(x, y) = \tilde{\rho}_0(x, y) + |d(x, A) - d(y, A)|$. \square

Now we can prove Lemma A.1 to finish the proof of Proposition 2.4.

Proof of Lemma A.1. Define $\rho: A \times A \rightarrow [0, \infty)$ by $\rho(x, y) = d(f(x), f(y))$. Then ρ is a uniformly continuous pseudometric on A . By Proposition A.5, for some $\varepsilon > 0$, we can extend ρ to a uniformly continuous pseudometric $\tilde{\rho}$ on $N = N(A, \varepsilon)$ such that A is closed in $(N, \tilde{\rho})$. Consider the quotient metric spaces $E = N/\tilde{\rho}$ and $B = A/\rho$. We can

⁵It should be noticed that Isbell [8] uses the term ANEU in the setting of uniform spaces, and Isbell's ANEU is more general when restricted to metric spaces.

consider B as a closed subset of E . The function $g: B \rightarrow X$, induced by f , is an isometric embedding. Then, by the Lemma A.2, we can embed X isometrically into $Z = E \sqcup X / \sim$ as a closed subset and g extends to an isometric embedding $\tilde{g}: E \rightarrow Z$. The desired extension $\tilde{f}: N \rightarrow Z$ can be defined as the composition $N \rightarrow E \xrightarrow{\tilde{g}} Z$. \square

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